MMAT5030 Notes 5

1 Wirtinger's Inequality

The following result will be used in the next section.

Theorem 5.1(Wirtinger's Inequality) For every $f \in R_{2\pi}$ satisfying $f' \in R_{2\pi}$,

$$\int_{-\pi}^{\pi} (f(x) - \overline{f})^2 dx \le \int_{-\pi}^{\pi} f'^2(x) dx \ ,$$

and equality holds if and only if $f(x) = a + b\cos x + c\sin x$ a.e. for some constants a, b and c. Here

$$\overline{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f$$

is the average or mean of f over $[-\pi, \pi]$.

Proof. Noting that $\overline{f} = a_0/2$,

$$f(x) - \overline{f} \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ .$$

By Parseval's identity,

$$\int_{-\pi}^{\pi} (f(x) - \overline{f})^2 dx = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \ ,$$

and

$$\int_{-\pi}^{\pi} (f(x) - \overline{f})'^2 dx = \int_{-\pi}^{\pi} f'(x)^2 dx = \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2).$$

Therefore, we have

$$\int_{-\pi}^{\pi} f'(x)^2 dx - \int_{-\pi}^{\pi} (f(x) - \overline{f})^2 dx = \pi \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2) \ ,$$

and the result follows.

This inequality is also known as Poincaré's Inequality.

2 The Isoperimetric Problem

The classical isoperimetric problem known to the ancient Greeks asserts that only the circle maximizes the enclosed area among all simple, closed curves of the same perimeter. In this section we will present a proof of this inequality by Fourier series. To formulate this geometric problem in analytic terms, we need to recall some facts from advanced calculus.

Indeed, a regular curve is a map γ from some interval $[a, b]$ to $\mathbb{R}^2$ such that x and y belong to $C^1[a, b]$ where $\gamma(t) = (x(t), y(t))$ and $x'^2(t) + y'^2(t) > 0$ for all $t \in [a, b]$. In the following a curve is always refereed to a regular curve. For such a curve, its length is defined to be

$$L[\gamma] = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt, \quad \gamma = (x, y).$$

A curve is closed if $\gamma(a) = \gamma(b)$ and simple if $\gamma(t) \neq \gamma(s),\ \forall t \neq s$ in $[a, b)$ (that is, it has no self-intersection). The length of a closed curve is called the perimeter of the curve.

When a closed, simple curve is given, the area it encloses is also fixed. Hence one should be able to express this enclosed area by a formula involving γ only. Indeed, this can be accomplished by the Green's theorem. Recalling that the Green's theorem states that for every pair of C^1-functions P and Q defined on the curve γ and the region enclosed by the curve, we have

$$\int_\gamma Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\Big)dxdy \ ,$$

where the left hand side is the line integral along γ and D is the domain enclosed by γ (see Fritzpatrick, p.543). Taking $P \equiv 0$ and $Q = x$, we obtain

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy = \iint_D 1 = \text{ area of } D,$$

so

$$A[\gamma] = \iint_D 1dxdy = \int_\gamma xdy = \int_a^b x(t)y'(t)dt \ .$$

The classical isoperimetric problem is: Among all simple, closed curves with a fixed perimeter, find the one whose enclosed area is the largest. We will see that the circle is the only solution to this problem.

To proceed further, let us recall the concept of reparametrization. Indeed, a curve γ_1 on $[a_1, b_1]$ is called a reparametrization of the curve γ on $[a, b]$ if there exists a continuously differentiable map ξ from $[a_1, b_1]$ to $[a, b]$ with non-vanishing derivative so that $\gamma_1(t) = \gamma(\xi(t)),\ \forall t \in [a_1, b_1]$. It can be verified that the length remains invariant under reparametrizations.

Another useful concept is the parametrization by arc-length. A curve $\gamma = (x, y)$ on $[a, b]$ is called in arc-length parametrization if $x'^2(t)+y'^2(t) = 1,\ \forall t \in [a, b]$. We know that every regular curve can be reparametrized in arc-length parametrization. Let $\gamma(t) = (x(t), y(t)),\ t \in [a, b]$, be a parametrization of a curve. We define a function φ by setting

$$\varphi(\tau) = \int_a^\tau \big(x'^2(t) + y'^2(t)\big)^{1/2}dt,$$

it is readily checked that φ is a continuously differentiable map from $[a, b]$ to $[0, L]$ with positive derivative, and $\gamma_1(s) = \gamma(\xi(s)),\ \xi = \varphi^{-1}$, is an arc-length reparametrization of γ on $[0, L]$ where L is the length of γ.

We now apply the Wirtinger's Inequality to give a proof of the classical isoperimetric problem.

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed, simple, regular curve bounding a region D. Without loss of generality we may assume that it is parametrized by arc-length. Assuming the perimeter of γ is equal to 2π, we want to find the region that encloses the maximal area. The perimeter is given by

$$L[\gamma] = \int_0^{2\pi} \sqrt{x'^2(s) + y'^2(s)}ds = 2\pi \ ,$$

and the area is given by

$$A[\gamma] = \int_0^{2\pi} x(s)y'(s)ds \ .$$

Extending γ_1 and γ_2 as 2π-periodic functions, we compute

$$\begin{aligned}
2A[\gamma] &= \int_{-\pi}^{\pi} 2x(s)y'(s)ds \\
&= \int_{-\pi}^{\pi} 2(x(s)-\overline{x})y'(s)ds \\
&\leq \int_{-\pi}^{\pi} (x(s)-\overline{x})^2 ds + \int_{-\pi}^{\pi} y^{'2}(s)ds \qquad (\text{by } 2ab \leq a^2+b^2) \\
&\leq \int_{-\pi}^{\pi} x^{'2}(s)ds + \int_{-\pi}^{\pi} y^{'2}(s)ds \qquad (\text{by Wirtinger's Inequality}) \\
&= \int_{-\pi}^{\pi} \left(x^{'2}(s)+y^{'2}(s)\right)ds \\
&= 2\pi, \qquad (\text{use } x'^2(s)+y'^2(s)=1)
\end{aligned}$$

whence $A[\gamma] \leq \pi$. We have shown that the enclosed area of a simple, closed, regular curve with perimeter 2π cannot exceed π. As π is the area of the unit circle, the unit circle solves the isoperimetric problem.

Now the uniqueness case. We need to examine the equality signs in our derivation. We observe that the second equality holds if and only if $a_n = b_n = 0$ for all $n \geq 2$ in the Fourier series of $x(s)$. So, $x(s) = a_0 + a_1 \cos s + b_1 \sin s$, or

$$x(s) = a_0 + r\cos(s-x_0), \quad a_0 = \overline{x} \ ,$$

where

$$r = \sqrt{a_1^2+b_1^2}, \ \cos x_0 = \frac{a_1}{r} \ .$$

(Note that $(a_1, b_1) \neq (0,0)$. For if $a_1 = b_1 = 0$, $x(s)$ is constant and $x^{'2} + y^{'2} = 1$ implies $y^{'2}(s) = \pm s + b$, and y can never be periodic.) Now we determine y. From the above calculation, when the first equality holds ($2ab = a^2+b^2$ means $a-b=0$),

$$x - \overline{x} - y' = 0.$$

So $y'(s) = x(s) - \overline{x} = r\cos(s-x_0)$, which gives

$$y(s) = r\sin(s-x_0) + c_0 \ , \quad c_0 \text{ constant.}$$

It follows that γ describes a circle of radius r centered at (a_0, c_0). Using the fact that the perimeter is 2π, we conclude that $r = 1$, so the maximum must be a unit circle.

For a general γ whose perimeter is not necessarily 2π, the "normalized curve" $c = 2\pi/L\gamma$ where L is the perimeter of c has perimeter 2π. The area enclosed by γ, A, is related to the area enclosed by c,A_0 by $A_0 = (2\pi/L)^2 A$. In the previous paragraph, we have shown that $A_0 \leq \pi$. Immediately, we deduce the isoperimetric inequality

$$A \leq \frac{L^2}{4\pi} \ ,$$

and the equality sign holds if and only if γ is a circle.

Summarizing, we have the following solution to the classical isoperimetric problem.

Theorem 5.2 (Isoperimetric Inequality) Among all closed, simple, regular curves of the same perimeter, only the circle encloses the largest area.

The isoperimetric inequality can also be interpreted as, among all regions which enclose the same area, only the circle has the shortest perimeter.